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Further Results on Sum Labelling of Split Graphs

J Rozario Gerard* and Lawrence Rozario Raj P†

Abstract

A *sum labelling* is a mapping λ from the vertices of G into the positive integers such that, for any two vertices $u, v \in V(G)$ with labels $\lambda(u)$ and $\lambda(v)$, respectively, uv is an edge iff $\lambda(u) + \lambda(v)$ is the label of another vertex in $V(G)$. Any graph supporting such a labelling is called a *sum graph*. It is necessary to add (as a disjoint union) a component to sum label a graph. This disconnected component is a set of isolated vertices known as *isolates* and the labelling scheme that requires the fewest isolates is termed *optimal*. The number of isolates required for a graph to support a sum labelling is known as the *sum number* of the graph. In this paper, we obtain *optimal sum labelling scheme* for path union of split graph of star, $K_{1,m} \odot \text{Spl}(P_n)$ and $K_{1,m} \odot \text{Spl}(K_{1,n})$.

Keywords: Sum labelling, sum graph, sum number, split graph, path union.

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1. Introduction

All the graphs considered here are simple, finite and undirected. For all terminologies and notations we follow Harary [1] and graph labelling as in [2]. Sum labelling of graphs was introduced by Harary [3] in 1990. Following definitions are useful for the present study.

* PG and Research Department of Mathematics, St Joseph's College of Arts and Science, Cuddalore, Tamil Nadu, India; rozario.gerard@yahoo.com.

† PG and Research Department of Mathematics, St. Joseph's College, Trichirappalli, Tamil Nadu, India; lawraj2006@yahoo.co.in.

Definition 1.1

Sum Labelling is a mapping λ from the vertices of G into the positive integers such that, for any two vertices $u, v \in V(G)$ with labels $\lambda(u)$ and $\lambda(v)$, respectively, uv is an edge iff $\lambda(u)+\lambda(v)$ is the label of another vertex in $V(G)$. Any graph supporting such a labelling is called a *sum graph*.

Definition 1.2

It is necessary to add (as a disjoint union) a component to sum label a graph. This disconnected component is a set of isolated vertices known as *isolates* and the labelling scheme that requires the fewest isolates is termed *optimal*.

Definition 1.3

The number of isolates required for a graph G to support a sum labelling is known as the *sum number* of the graph. It is denoted as $\sigma(G)$.

Definition 1.4

(Shiama [4]) For a graph G the split graph is obtained by adding to each vertex v , a new vertex v' such that v' is adjacent to every vertex that is adjacent to v in G . The resultant graph is called the *split graph* denoted by $\text{Spl}(G)$.

Definition 1.5

(Shee and Ho [5]) Let G_1, G_2, \dots, G_n , $n \geq 2$ be n copies of a fixed graph G . The graph obtained by adding an edge between G_i and G_{i+1} for $i=1, 2, \dots, n-1$ is called *path union* of G .

2. Sum Labelling for Split Graphs

In [6], Gerard *et al.* has proved that split graph of path, star are sum graph with sum number 1 and bi-star is sum graph with sum number 2.

Theorem 2.1

Path union of split graph of star $K_{1,n}$ is a sum graph with sum number 1.

Proof

Consider a star $K_{1,n}$ with $(n+1)$ vertices. Let G be the split graph of star, $\text{Spl}(K_{1,n})$. Let G^* be the path union of m copies of G . Let $v_1, v_{11}, v_{12}, \dots, v_{1n}, v_2, v_{21}, v_{22}, \dots, v_{2n}, \dots, v_m, v_{m1}, v_{m2}, \dots, v_{mn}$ be the vertices of m copies of the star $K_{1,n}$. Let $u_1, u_{11}, u_{12}, \dots, u_{1n}, u_2, u_{21}, u_{22}, \dots, u_{2n}, \dots, u_m, u_{m1}, u_{m2}, \dots, u_{mn}$ be the vertices corresponding to $v_1, v_{11}, v_{12}, \dots, v_{1n}, v_2, v_{21}, v_{22}, \dots, v_{2n}, \dots, v_m, v_{m1}, v_{m2}, \dots, v_{mn}$ of m copies of the star $K_{1,n}$ which are added, to obtain the split graph of m copies of star. G^* has $3nm$ vertices and $3nm + (m - 1)$ edges. Let x be the isolated vertex.

Define $f: V(G^*) \rightarrow \mathbb{N}$

$$\begin{aligned} f(v_1) &= 1 & f(v_2) &= 2 \\ f(v_i) &= f(v_{(i-1)}) + f(v_{(i-2)}) \text{ for } 3 \leq i \leq m \\ f(v_{11}) &= f(v_m) + f(v_{(m-1)}) \\ \text{for } 1 \leq i \leq m & \\ \left\{ \begin{array}{l} f(v_{ij}) = f(v_{i(j-1)}) + f(v_i) \text{ for } 2 \leq j \leq n \\ f(u_i) = f(v_{in}) + f(v_i) \\ f(u_{i1}) = f(u_i) + f(v_{i1}) \\ f(u_{ij}) = f(u_{i(j-1)}) + f(v_i) \text{ for } 2 \leq j \leq n \\ f(v_{(i+1)1}) = f(u_{in}) + f(v_i) \text{ if } i \neq m \\ f(x) = f(u_{mn}) + f(v_m) \end{array} \right. \end{aligned}$$

Thus, Path union of Split graph of star $K_{1,n}$ is a sum graph with sum number 1.

Theorem 2.2

$K_{1,m} \odot \text{Spl}(P_n)$ is a sum graph with sum number 1.

Proof

Let c, v_1, v_2, \dots, v_m be the vertices of $K_{1,m}$ where c is the centre of the star. Let $v_{11}, v_{12}, \dots, v_{1n}, u_{11}, u_{12}, \dots, u_{1n}, v_{21}, v_{22}, \dots, v_{2n}, u_{21}, u_{22}, \dots, u_{2n}, \dots, v_{m1}, v_{m2}, \dots, v_{mn}, u_{m1}, u_{m2}, \dots, u_{mn}$ be the vertices of the m copies of the split graph of path P_n . The vertices $v_{11}, v_{21}, \dots, v_{m1}$ are attached to the vertices v_1, v_2, \dots, v_m respectively. Let $G = K_{1,m} \odot \text{Spl}(P_n)$. Therefore the vertex set of G , $V(G) = \{c, v_{11}, v_{12}, \dots, v_{1n}, u_{11}, u_{12}, \dots, u_{1n}, v_{21}, v_{22}, \dots, v_{2n}, u_{21}, u_{22}, \dots, u_{2n}, \dots, v_{m1}, v_{m2}, \dots, v_{mn}, u_{m1}, u_{m2}, \dots, u_{mn}\}$. G has $2nm + 1$ vertices and $3n(n - 1) + m$ edges. Let x be the isolated vertex.

Define $f: V(G) \rightarrow \mathbb{N}$

$$\begin{aligned}
 f(c) &= 1 & f(v_{11}) &= 2 \\
 & \text{for } 1 \leq i \leq m \\
 \left\{ \begin{aligned}
 f(v_{i2}) &= f(v_{i1}) + 1 \\
 f(v_{ij}) &= f(v_{i(j-1)}) + f(v_{i(j-2)}) \text{ for } 3 \leq j \leq n \\
 f(u_{i1}) &= f(v_{i(n-1)}) + f(v_{in}) \\
 f(u_{i2}) &= f(u_{i1}) + 1 \\
 f(u_{ij}) &= f(u_{i(j-1)}) + f(v_{i(j-2)}) \text{ for } 3 \leq j \leq n \\
 f(v_{(i+1)1}) &= f(u_{in}) + f(v_{i(n-1)}) \text{ if } i \neq m \\
 f(x) &= f(u_{mn}) + f(v_{m(n-1)})
 \end{aligned} \right.
 \end{aligned}$$

Hence, $K_{1,m} \odot \text{Spl}(P_n)$ is a sum graph with sum number 1.

Theorem 2.3

$K_{1,m} \odot \text{Spl}(K_{1,n})$ is a sum graph with sum number 1.

Proof

Let c, v_1, v_2, \dots, v_m be the vertices of $K_{1,m}$ where c is the centre of the star. Let $c_1, v_{11}, v_{12}, \dots, v_{1n}, u_1, u_{11}, u_{12}, \dots, u_{1n}, c_2, v_{21}, v_{22}, \dots, v_{2n}, u_2, u_{21}, u_{22}, \dots, u_{2n}, \dots, c_m, v_{m1}, v_{m2}, \dots, v_{mn}, u_m, u_{m1}, u_{m2}, \dots, u_{mn}$ be the vertices of the m copies of the split graph of star $K_{1,n}$. The vertices c_1, c_2, \dots, c_m are attached to the vertices v_1, v_2, \dots, v_m of $K_{1,n}$ respectively. Let $G = K_{1,m} \odot \text{Spl}(K_{1,n})$. Therefore the vertex set of G , $V(G) = \{c, c_1, v_{11}, v_{12}, \dots, v_{1n}, u_1, u_{11}, u_{12}, \dots, u_{1n}, c_2, v_{21}, v_{22}, \dots, v_{2n}, u_2, u_{21}, u_{22}, \dots, u_{2n}, \dots, c_m, v_{m1}, v_{m2}, \dots, v_{mn}, u_m, u_{m1}, u_{m2}, \dots, u_{mn}\}$. G has $2nm + 1$ vertices and $3n(n-1) + m$ edges. Let x be the isolated vertex.

Define $f: V(G) \rightarrow \mathbb{N}$

$$\begin{aligned}
 f(c) &= 1 & f(c_1) &= 2 \\
 f(c_i) &= f(c_{(i-1)}) + 1 & \text{for } 2 \leq i \leq m \\
 f(v_{11}) &= f(c_m) + 1 \\
 & \text{for } 1 \leq i \leq m \\
 \left\{ \begin{aligned}
 f(v_{ij}) &= f(v_{i(j-1)}) + f(v_i) \text{ for } 2 \leq j \leq n \\
 f(u_i) &= f(v_{in}) + f(c_i) \\
 f(u_{i1}) &= f(u_i) + f(v_{i1}) \\
 f(u_{ij}) &= f(u_{i(j-1)}) + f(c_i) \text{ for } 2 \leq j \leq n \\
 f(v_{(i+1)1}) &= f(u_{in}) + f(c_i) \text{ if } i \neq m \\
 f(x) &= f(u_{mn}) + f(c_m)
 \end{aligned} \right.
 \end{aligned}$$

Thus, $K_{1,m} \odot \text{Spl}(K_{1,n})$ is a sum graph with sum number 1.

Illustration: Sum labelling for path union of split graph of star $K_{1,n}$ is given in figure 2.1

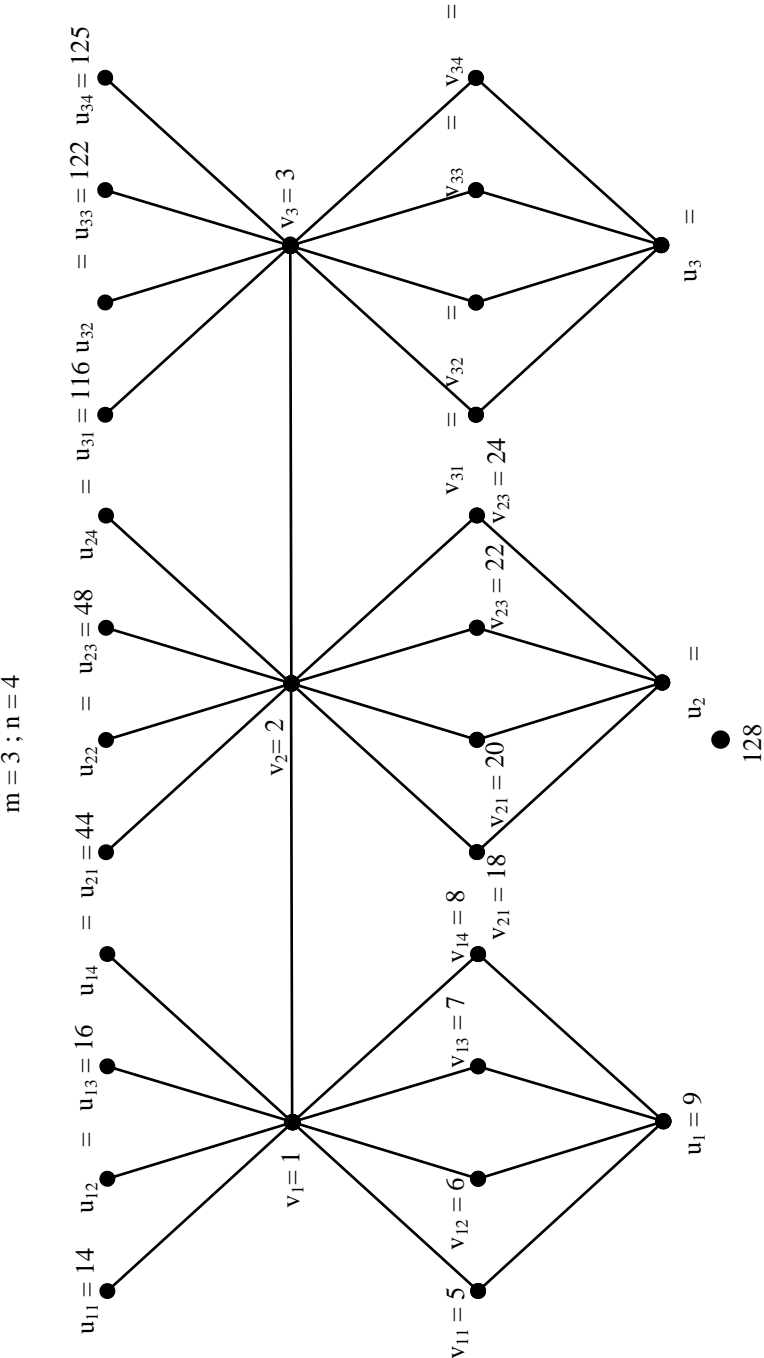


Fig. 2.1

$K_{1,m} \odot Spl(K_{1,n})$ $K_{1,m} \odot Spl(K_{1,n})$ is obtained by attaching a copy of $Spl(K_{1,n})$ to each pendent vertex of $K_{1,m}$.

Illustration Sum labelling for $K_{1,m} \odot Spl(K_{1,n})$ is given in figure 2.3

$m = 3; n = 2$

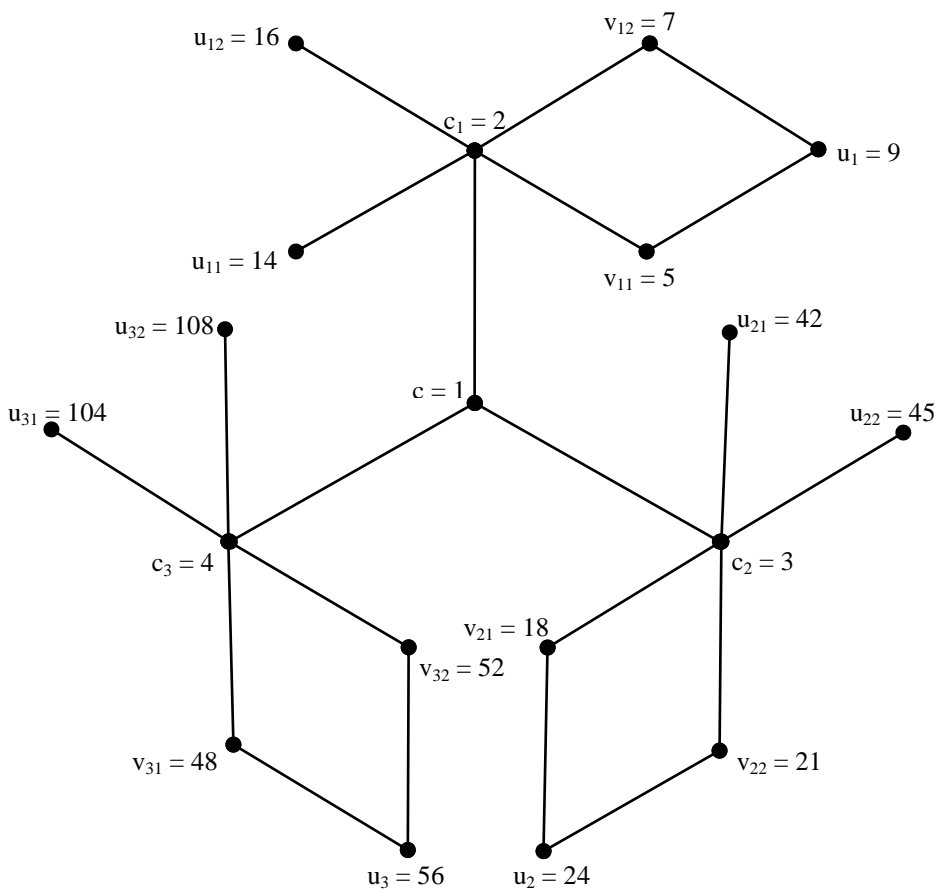


Fig 2.3

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